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BERNSTEIN POLYNOMIAL APPROACH FOR SOLUTION OF HIGHER-ORDER MIXED LINEAR FREDHOLM INTEGRO-DIFFERENTIAL-DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. Bernstein collocation matrix method is presented to solve m-th order linear Fredholm integro-differential-difference equations subjected to mixed conditions. The methodology is based on approximation by the truncated Bernstein series, which converts the given equation and the conditions into a system of linear algebraic equations with Bernstein coefficients. By solving the arising system, the Bernstein coefficients of the solution can be obtained. The method is also valid for any combination of differential, difference, differential-difference and Fredholm integral equations. The applicability and validity of the proposed scheme is demonstrated by numerical experiments and comparative analysis of the results is given too.

Keywords: mixed linear Fredholm integro-differential-difference equations, Bernstein polynomials and series, collocation points.

AMS Subject Classification: 34k28, 34k10, 45A05, 45B05, 45J05, 39A99.

1. Introduction

Mixed integro-differential-difference equations have become the focus for researchers. These equations are conventionally categorized into the Fredholm integro-differential-difference equations and Volterra integro-differential-difference equations.

These equations are frequently used in many applied domains including engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc [3, 6, 11-13, 15, 17, 21, 22, 24, 27, 29, 33-35, 37, 38, 40, 47-49, 51, 57]. The mathematicians and physicists have devoted considerable effort aimed at the numerical solutions of the integro-differential-difference equations.

For instance, successive approximations, Adomian decomposition, Chebyshev and Taylor collocation methods, Haar Wavelet method, Taylor, Tau, Legendre and Walsh series methods, Bessel matrix method, variational iteration method, Legendre wavelet method, Sine-Cosine wavelets, finite difference method, rationalized Haar functions, Cas wavelet, differential transform method, Homotopy perturbation method, etc [1, 5, 7, 8, 15, 20, 26, 36, 40, 43-46, 48, 54-56].

These equations befall repeatedly as a model in mathematical biology and the physical sciences [17, 39]. The integro-difference equations given in [39, pp.304, 320] describe the efflux of gas from the open end of a tube and the integro-differential-difference equation given by the integral equation of Palm arises in queuing theory [39, p.304]. The integro-differential equation given in [57] is a vivid example of one-dimensional viscoelasticity and also a model for circulating fuel in nuclear reactors [17].

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Boundary value problems involving integro-differential-difference equations in studying variational elements of control theory, when encountering a complicated problem using the time delays effect [7, 8] and signal transmission [39], biological problems for determining expected time for generation of action potentials in nerve cells [1, 2, 57]. This is applied in modeling activation of a neuron [7], on epidemics and population [39], in two-body problems of physical systems whose acceleration depends upon its velocity and its position at earlier instants, and in formulation of biological reaction phenomena to X-rays [17, 44, 39].

The Monte Carlo and direct methods are applied by Farnoosh and Ebrahimi and Asady et al., respectively [14, 2]. Taylor and Chebyshev matrix methods have been used by Sezer et al. since the early 1990s to solve the linear differential, Fredholm integral, Fredholm integro-differential, difference, integro-difference and systems of integro-differential equations [44, 30, 23, 41, 42, 16, 18].

The major emphasis of the study is on the Fredholm integro-differential-difference equations, where as all the algorithms employed in the paper are used for the Volterra integro-differential-difference equation with minor modifications. Although integro-differential-difference equations are crucially importance, nonetheless, are hard to solve analytically as well as numerically.

The basic idea of this study is to developed the numerical solution to the m -th order linear Fredholm integro-differential-difference equation with variable coefficients

$$\begin{aligned}\sum_{k=0}^{m}\sum_{j=0}^{r} p_{kj}(x)\, y^{(k)}(\mu_{kj}x+\tau_{kj}) = g(x)+ \\ \lambda \int_0^1 \sum_{p=0}^{q}\sum_{l=0}^{s} \kappa_{pl}(x,\,t)\, y^{(p)}(\gamma_{pl}\,t+\delta_{pl})\,dt, \quad 0\le x,\,t\le 1,\end{aligned} \tag{1}$$

where $q \le m$, under the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{\xi=0}^{f} a_{\xi L}^{k}\, y^{(k)}(\eta_\xi) = \lambda_L, \quad L=0,\,\ldots,\,m-1. \tag{2}$$

Here $\{p_{kj}(x)\}_{k=0,\,j=0}^{m,\,r}$, $g(x)\in L^2[0,\,1]$ and $\{\kappa_{pl}(x,\,t)\}_{p=0,\,l=0}^{q,\,s} \in L^2([0,\,1]\times[0,\,1])$ are known functions, $y(x)$ is the unknown function to be determined and the real coefficients $\{\mu_{kj},\,\tau_{kj}\}_{k=0,\,j=0}^{m,\,r}$, $\{\gamma_{pl},\,\delta_{pl}\}_{p=0,\,l=0}^{q,\,s}$, $\left\{a_{\xi L}^{k},\,\lambda_L\right\}_{\xi=0,\,k=0,\,L=0}^{f,\,m-1,\,m-1}$ and λ are appropriate constants. Note that η_ξ, $\xi = 0,\,\ldots,\,f$ are given points in the spatial domain of the problem. The aim is therefore, to find an approximate solution of Eq. (1), expressed in the following form:

$$y(x)\cong y_{n+1}(x)=\sum_{i=0}^{n} c_i\,B_{i,\,n}(x), \quad 0\le x\le 1, \tag{3}$$

where, for $n \ge 1$ the functions $\{B_{i,\,n}(x)\}_{i=0}^{n}$ denotes the Bernstein basis polynomials (B-polynomials) of n -th degree, as defined by [4]:

$$B_{i,\,n}(x)=\binom{n}{i}x^i(1-x)^{n-i}, \qquad 0\le x\le 1, \;\; i=0,\,1,\,\ldots,\,n, \tag{4}$$

c_i, $i = 0,\,\ldots,\,n$, are the unknown Bernstein coefficients that ought to be determined and n represents any positive integer in such a manner that $n \ge m$. The collection points are selected to obtain an approximate solution in the form (3) of the problem (1) and (2), as expressed and defined by:

$$x_i=\frac{i}{n}, \quad i=0,\,\ldots,\,n. \tag{5}$$

The Bernstein polynomials are applied to determine the approximate solutions of differential, linear and nonlinear Fredholm integral and integro-differential equations and nonlinear Volterra-Fredholm-Hammerstein integral equations [53, 32, 10, 31, 28].

The paper is structured to encompass the introduction, followed by the fundamental matrix relations related to the m -th order linear Fredholm integro-differential-difference equation with variable coefficients. Section 3 deals with the solution methods for these equations. Section 4, provides an explanation on the accuracy of solving the linear Fredholm integro-differential-difference equation with variable coefficients. Finally, section 5, provides few numerical examples, followed by overall conclusions of the study.

2. Fundamental matrix relations

By simplification of the Eq. (1), a new relation can be derived as follows

$$D(x) = g(x) + \lambda\, I_F(x), \tag{6}$$

where, the differential-difference part

$$D(x) = \sum_{k=0}^{m}\sum_{j=0}^{r} p_{kj}(x)\, y^{(k)}(\mu_{kj}x + \tau_{kj}), \tag{7}$$

and the Fredholm integral part can be expressed in the following form:

$$I_F(x) = \int_0^1 \sum_{p=0}^{q}\sum_{l=0}^{s} \kappa_{pl}(x,\, t)\, y^{(p)}(\gamma_{pl}\, t + \delta_{pl})\, dt. \tag{8}$$

These parts and the mixed conditions (2) are converting to the matrix forms in the following sections.

The approximate solution $y(x)$ and its k -th derivative $y^{(k)}(x)$ as defined by a truncated Bernstein series (3) are considered. The truncated series (3) are expressed in the matrix form as follows:

$$\begin{gathered} y(x) \cong B(x)\, C, \\ y^{(k)}(x) \cong B^{(k)}(x)\, C, \end{gathered} \tag{9}$$

where,

$$\begin{gathered} B(x) = \left[B_{0,n}(x),\, B_{1,n}(x),\, \dots,\, B_{n,n}(x)\right], \\ C = \left[c_0,\, c_1,\, \dots,\, c_n\right]^T. \end{gathered} \tag{10}$$

By applying the binomial expansion of $(1-x)^{n-i}$, the following relation is derived at:

$$B_{i,n}(x) = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} x^{i+k}. \tag{11}$$

The corresponding matrix relation can be found by the following equation:

$$B(x) = T_n(x)\, A \;\; or \;\; B^T(x) = A^T T_n^T(x), \tag{12}$$

where

$$T_n(x) = \left[1,\, x,\, x^2,\, \dots,\, x^n\right], \tag{13}$$

and

$$A = \begin{bmatrix} (-1)^0\binom{n}{0} & 0 & 0 & \dots & 0 \\ (-1)^1\binom{n}{0}\binom{n-0}{1} & (-1)^0\binom{n}{1} & 0 & \dots & 0 \\ (-1)^2\binom{n}{0}\binom{n-0}{2} & (-1)^1\binom{n}{1}\binom{n-1}{1} & (-1)^0\binom{n}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} & (-1)^{n-2}\binom{n}{2}\binom{n-2}{n-2} & \dots & (-1)^0\binom{n}{n} \end{bmatrix}. \tag{14}$$

By substituting Eq. (12) into Eq. (9), the matrix forms for $y(x)$ and $y^{(k)}(x)$ are derived in the form of following equations:

$$y(x) \cong T_n(x)\, A\, C, \tag{15}$$

and

$$y^{(k)}(x) \cong T_n^{(k)}(x)\, A\, C. \tag{16}$$

There is a relation between the vector $T_n(x)$ and its derivative $T_n^{(1)}(x)$ as follow:

$$\begin{aligned} T_n^{(1)}(x) &= \left[1,\, x,\, x^2,\, \ldots,\, x^n\right]^{(1)} = \left[0,\, 1,\, 2\,x,\, \ldots,\, n\,x^{n-1}\right] \\ &= \left[1,\, x,\, x^2,\, \ldots,\, x^n\right] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ &= T_n(x)\, D_{T_n}, \end{aligned} \tag{17}$$

where

$$D_{T_n} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{18}$$

The procedure for arriving at the vector $T_n^{(k)}(x)$ is used in terms of the vector $T_n(x)$ as follows:

$$\begin{aligned} T_n^{(1)}(x) &= T_n(x)\,(D_{T_n})^1, \\ T_n^{(2)}(x) &= T_n^{(1)}(x) D_{T_n} = (T_n(x)\, D_{T_n})\; D_{T_n} = T_n(x)\,(D_{T_n})^2, \\ T_n^{(3)}(x) &= T_n^{(2)}(x) D_{T_n} = \left(T_n(x)\,(D_{T_n})^2\right)\; D_{T_n} = T_n(x)\,(D_{T_n})^3, \\ &\vdots \\ T_n^{(k)}(x) &= T_n^{(k-1)}(x) D_{T_n} = \left(T_n(x)\,(D_{T_n})^{k-1}\right)\; D_{T_n} = T_n(x)\,(D_{T_n})^k. \end{aligned} \tag{19}$$

The matrix form of the k -th derivatives of function $y(x)$ is obtained, by replacing the matrix Eq. (19) into Eq. (16), as follows

$$y^{(k)}(x) \cong T_n(x)\,(D_{T_n})^k\, A\, C. \tag{20}$$

2.1. **Matrix relations for the differential-difference part.** Substitution of the quantities $\mu_{kj}x + \tau_{kj}$ for the x in Eq. (20) the matrix form can therefore, be obtained by the following equation:

$$y^{(k)}(\mu_{kj}x + \tau_{kj}) \cong T_n(\mu_{kj}x + \tau_{kj})\,(D_{T_n})^k A\, C, \tag{21}$$

which according to Eq. (13), the following relation can be derived at:

$$T_n(\mu_{kj}x + \tau_{kj}) = \Big[1,\, \mu_{kj}x + \tau_{kj},\, (\mu_{kj}x + \tau_{kj})^2,\, \ldots,\, (\mu_{kj}x + \tau_{kj})^n\Big]. \tag{22}$$

All elements of vector $T_n(\mu_{kj}x + \tau_{kj})$ in terms of $\left\{x^i\right\}_{i=0}^n$ are approximated, in order to obtain the vector $T_n(\mu_{kj}x + \tau_{kj})$ in terms of the vector $T_n(x)$. Application of the binomial expansion of $(\mu_{kj}x+\tau_{kj})^R$, $R=0,\,1,\,\ldots,\,n$, makes it possible to derive the following equations:

$$\begin{aligned}
1 &= \tbinom{0}{0}\mu_{kj}^0\,\tau_{kj}^0,\\
\mu_{kj}x+\tau_{kj} &= \tbinom{1}{0}\mu_{kj}^0\,\tau_{kj}^1+\tbinom{1}{1}\mu_{kj}^1\,\tau_{kj}^0\,x,\\
(\mu_{kj}x+\tau_{kj})^2 &= \tbinom{2}{0}\mu_{kj}^0\,\tau_{kj}^2+\tbinom{2}{1}\mu_{kj}^1\,\tau_{kj}^1\,x+\tbinom{2}{2}\mu_{kj}^2\,\tau_{kj}^0\,x^2\,, \qquad (23)\\
&\vdots\\
(\mu_{kj}x+\tau_{kj})^n &= \tbinom{n}{0}\mu_{kj}^0\,\tau_{kj}^n+\tbinom{n}{1}\mu_{kj}^1\,\tau_{kj}^{n-1}\,x+\tbinom{n}{2}\mu_{kj}^2\,\tau_{kj}^{n-2}\,x^2+\cdots+\tbinom{n}{n}\mu_{kj}^n\,\tau_{kj}^0\,x^n.
\end{aligned} \tag{23}$$

Therefore, by defining

$$Z(\mu_{kj},\,\tau_{kj})=\begin{bmatrix}
\binom{0}{0}\mu_{kj}^0\,\tau_{kj}^0 & \binom{1}{0}\mu_{kj}^0\,\tau_{kj}^1 & \binom{2}{0}\mu_{kj}^0\,\tau_{kj}^2 & \cdots & \binom{n}{0}\mu_{kj}^0\,\tau_{kj}^n\\
0 & \binom{1}{1}\mu_{kj}^1\,\tau_{kj}^0 & \binom{2}{1}\mu_{kj}^1\,\tau_{kj}^1 & \cdots & \binom{n}{1}\mu_{kj}^1\,\tau_{kj}^{n-1}\\
0 & 0 & \binom{2}{2}\mu_{kj}^2\,\tau_{kj}^0 & \cdots & \binom{n}{2}\mu_{kj}^2\,\tau_{kj}^{n-2}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & \binom{n}{n}\mu_{kj}^n\,\tau_{kj}^0
\end{bmatrix}, \tag{24}$$

the relation between vectors $T_n(\mu_{kj}x+\tau_{kj})$ and $T_n(x)$ is

$$T_n(\mu_{kj}x+\tau_{kj})=T_n(x)\,Z(\mu_{kj},\,\tau_{kj}). \tag{25}$$

By substituting the matrix form (25) into Eq. (21), the following matrix relation can be derived at:

$$y^{(k)}(\mu_{kj}x+\tau_{kj})\cong T_n(x)\,Z(\mu_{kj},\,\tau_{kj})(D_{T_n})^k A\,C. \tag{26}$$

Thus, the matrix representation of the differential-difference part can be expressed in the following form:

$$D(x)=\sum_{k=0}^{m}\sum_{j=0}^{r}P_{kj}(x)\,T_n(x)\,Z(\mu_{kj},\,\tau_{kj})(D_{T_n})^k A\,C. \tag{27}$$

2.2. **Matrix relations for the Fredholm integral part.** The matrix relation for Fredholm integral part $I_F(x)$ in Eq. (6) can now be found. The kernel functions $\{\kappa_{pl}(x,\,t)\}_{p=0,\,l=0}^{q,\,s}$ can be expanded in B-Polynomials as follows:

$$\kappa_{pl}(x,\,t)\cong\sum_{i=0}^{n}\sum_{\sigma=0}^{n}k_{i\sigma}^{pl}B_{i,\,n}(x)\,B_{\sigma,\,n}(t). \tag{28}$$

Then the matrix representation of the kernel functions $\{\kappa_{pl}(x,\,t)\}_{p=0,\,l=0}^{q,\,s}$ is given expressed by the following relation:

$$\kappa_{pl}(x,\,t)\cong B(x)K_{pl}B^T(t), \tag{29}$$

where

$$K_{pl}=\left[k_{i\sigma}^{pl}\right],\quad i,\,\sigma=0,\,\ldots,\,n, \tag{30}$$

and the elements of K_{pl} are given by

$$k_{i\sigma}^{pl}=\frac{\langle B_{i,\,n}(x),\,\langle\kappa_{pl}(x,\,t),\,B_{\sigma,\,n}(t)\rangle\rangle}{\langle B_{i,\,n}(x),\,B_{i,\,n}(x)\rangle\,\langle B_{\sigma,\,n}(t),\,B_{\sigma,\,n}(t)\rangle},\quad i,\,\sigma=0,\,\ldots,\,n. \tag{31}$$

Substituting the Eq. (12) into Eq. (29) in order to derive

$$\kappa_{pl}(x,\,t)\cong T_n(x)\,A\,K_{pl}\,A^T\,T_n^T(t). \tag{32}$$

By substituting the matrix forms (32) and (26) in accordance with the functions $\{\kappa_{pl}(x,\,t)\}_{p=0,\,l=0}^{q,\,s}$ and $y^{(p)}(\gamma_{pl}\,t+\delta_{pl})$ into the Fredholm integral part the fundamental matrix equations are obtained as follows:

$$\begin{aligned} I_F(x) &\cong \int_0^1 \sum_{p=0}^{q}\sum_{l=0}^{s} T_n(x)\,A\,K_{pl}\,A^T\,T_n^T(t)\,T_n(t)\,Z(\gamma_{pl},\,\delta_{pl})(D_{T_n})^p A\,C\,dt \\ &= T_n(x)\sum_{p=0}^{q}\sum_{l=0}^{s} A\,K_{pl}\,A^T\left\{\int_0^1 T_n^T(t)\,T_n(t)\,dt\right\}\,Z(\gamma_{pl},\,\delta_{pl})(D_{T_n})^p A\,C. \end{aligned} \tag{33}$$

The matrix representation of the Fredholm integral part can be expressed as:

$$I_F(x) \;=\; T_n(x)\sum_{p=0}^{q}\sum_{l=0}^{s} A\,K_{pl}\,A^T\,H\;Z(\gamma_{pl},\,\delta_{pl})(D_{T_n})^p A\,C\,, \tag{34}$$

in which H is a well-known Hilbert matrix

$$H=\int_0^1 T_n^T(t)\,T_n(t)\,dt=\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix}. \tag{35}$$

2.3. **Matrix relations for the mixed conditions.** The Eq.(20) can be used to obtain the corresponding matrix form for the m mixed conditions (2), according to which the followings equation can be expressed:

$$\sum_{k=0}^{m-1}\sum_{\xi=0}^{f}\left[a_{\xi L}^{k}T_n(\eta_\xi)\right]\,(D_{T_n})^k\,A\,C\;=\lambda_L,\quad L=0,\,\ldots,\,m-1. \tag{36}$$

The matrix form for conditions (2) can therefore be written by the following equation:

$$U_L\,C=[\lambda_L]\quad or\quad [U_L;\,\lambda_L]\;,\;\; L=0,\,\ldots,\,m-1, \tag{37}$$

where

$$U_L=\sum_{k=0}^{m-1}\sum_{\xi=0}^{f}\left[a_{\xi L}^{k}T_n(\eta_\xi)\right]\,(D_{T_n})^k\,A\;=[u_{L\,0},\,u_{L\,1},\ldots,\,u_{L\,n}]\,. \tag{38}$$

3. Solution method

The fundamental matrix equation corresponding to Eq. (1) can now be constructed. For this reason, the Eq. (27) and Eq. (34) is substituted in the Eq. (1) in order to at the following matrix equation

$$\begin{aligned} &\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x)\,T_n(x)\,Z(\mu_{kj},\,\tau_{kj})(D_{T_n})^k A\,C=g(x) \\ &+\lambda\,T_n(x)\sum_{p=0}^{q}\sum_{l=0}^{s} A\,K_{pl}\,A^T\,H\;Z(\gamma_{pl},\,\delta_{pl})(D_{T_n})^p A\,C\,. \end{aligned} \tag{39}$$

By substituting the collocation points (5) into Eq. (39), the system of matrix equations are derived at as follows

$$\begin{aligned} &\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}(x_i)\,T_n(x_i)\,Z(\mu_{kj},\,\tau_{kj})(D_{T_n})^k A\,C=g(x_i) \\ &+\lambda\,T_n(x_i)\sum_{p=0}^{q}\sum_{l=0}^{s} A\,K_{pl}\,A^T\,H\;Z(\gamma_{pl},\,\delta_{pl})(D_{T_n})^p A\,C\,. \end{aligned} \tag{40}$$

Eq. (40) can be summarized and expressed in the following form

$$\left\{\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}\, T\, Z(\mu_{kj},\, \tau_{kj})(D_{T_n})^k - \lambda\, T \sum_{p=0}^{q}\sum_{l=0}^{s} A\, K_{pl}\, A^T\, H\;\, Z(\gamma_{pl},\, \delta_{pl})(D_{T_n})^p\right\} AC = G, \quad (41)$$

where

$$P_{kj} = \begin{bmatrix} P_{kj}(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_{kj}(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_{kj}(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_{kj}(x_n) \end{bmatrix},$$

$$T = \begin{bmatrix} T_n(x_0) \\ T_n(x_1) \\ T_n(x_2) \\ \vdots \\ T_n(x_n) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}, \quad (42)$$

$$G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}.$$

Let

$$W = \left\{\sum_{k=0}^{m}\sum_{j=0}^{r} P_{kj}\, T\, Z(\mu_{kj},\, \tau_{kj})(D_{T_n})^k - \lambda\, T \sum_{p=0}^{q}\sum_{l=0}^{s} A\, K_{pl}\, A^T\, H\;\, Z(\mu_{pl},\, \tau_{pl})(D_{T_n})^p\right\} A. \quad (43)$$

Hence, the fundamental matrix equation for Eq. (1) can be reduced to show the relation expressed in the following form:

$$W\,C = G \;\; or \;\; [W; G], \quad (44)$$

which corresponds to a system of $n+1$ linear algebraic equations with the $n+1$ unknown Bernstein coefficients $c_0, \ldots, c_n$. Consequently, to find the approximate solution of Eq. (1) under mixed conditions (2), the rows matrices (37) is replaced by the rows of matrices (44), in order to yield the new augmented matrix $\bar{W}C = \bar{G}$. For convenience if the last m rows of the matrix (44) are replaced, the augmented matrix of the above system is as follows:

$$[\bar{W}; \bar{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n}; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1n}; & g(x_1) \\ \vdots & \vdots & & \vdots & \vdots \\ w_{n-m,\,0} & w_{n-m,\,1} & \cdots & w_{n-m,\,n}; & g(x_{n-m}) \\ u_{00} & u_{01} & \cdots & u_{0n}; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1n}; & \lambda_1 \\ \vdots & \vdots & & \vdots & \vdots \\ u_{m-1,\,0} & u_{m-1,\,1} & \cdots & u_{m-1,\,n}; & \lambda_{m-1} \end{bmatrix}. \quad (45)$$

However, we do not have to replace the last rows. If $rank(\bar{W}) = rank([\bar{W};\, \bar{G}]) = n+1$, then the relation can be expressed as:

$$C = (\bar{W})^{-1}\,(\bar{G}). \tag{46}$$

The matrix C (thereby the coefficients $c_0,\, \ldots,\, c_n$) is uniquely determined by Eq. (46). Thus the m -th order linear Fredholm integro-differential-difference equation with variable coefficients (1) under the mixed conditions (2) has a unique solution, provided by the truncated Bernstein series (3). Under circumstances where, $\det(\bar{W}) = 0$ and $rank(\bar{W}) = rank([\bar{W};\, \bar{G}]) < n+1$, any particular or general solution can be arrived at. If $rank(\bar{W}) \neq rank([\bar{W};\, \bar{G}]) < n+1$, then no solution would be contemplated. When conditions are not given, a general solution can be obtained by taking rows zero of the augmented matrix $[\bar{W};\, \bar{G}]$.

4. Accuracy of solution

If the exact solution is known, then the error function would be the difference between approximate solution $y_{n+1}(x)$ and the exact solution $y(x)$ defined by $E_{n+1}(x) = |y(x) - y_{n+1}(x)|$. Otherwise, it has to be estimated as follows [15, 9, 52]. Since the truncated Bernstein series (3) is an approximate solution of Eq. (1), and where the solution $y_{n+1}(x)$ and its derivatives are substituted in Eq. (1), the resulting equation should therefore be approximately satisfied; that is, for $x = x_\vartheta \in [0,\, 1];\ \vartheta = 0,\, 1,\, 2,\, \ldots$

$$E(x_\vartheta) = |D(x_\vartheta) - g(x_\vartheta) - \lambda\, I_F(x_\vartheta)| \cong 0, \tag{47}$$

or

$$E(x_\vartheta) \leq 10^{-\theta_\vartheta} \ \ (\theta_\vartheta \ is\, any\, positive\, \text{integer}). \tag{48}$$

If

$$\max\, 10^{-\theta_\vartheta} = 10^{-\theta} \ \ (\theta \ is\, any\, positive\, \text{integer}), \tag{49}$$

is prescribed, then the truncation limit n is increased until the difference $E(x_\vartheta)$ at each points x_ϑ becomes smaller than the prescribed $10^{-\theta}$. The error function can thus, be estimated by the following relation:

$$E_{n+1}(x) = D(x) - g(x) - \lambda\, I_F(x). \tag{50}$$

If $E_{n+1}(x) \to 0$, when n is sufficiently large enough, then the error decreases.

5. Illustrative examples

The method presented here is applied to solve six cases, the computations of which are carried out by application of Matlab 7.1 on a PC computer. When the problem is defined in a finite range $[a,\, b]$, by using the linear transformation

$$x = (b-a)\,\rho + a, \tag{51}$$

the range of which can then be converted to the range $[0,\, 1]$. The absolute errors in Tables are the values of $|y(x) - y_{n+1}(x)|$ at the selected points of the interval $[0,\, 1]$.

Example 1. Consider the second order linear Fredholm integro-differential-difference equation given in [25] by

$$\begin{array}{l} y''(x) + y''(x-1) + 2x\, y'(x-2) = \\ 4x^2 - 15x + 4 + 12\int_0^1 x\, t\, y(t)\, dt, \quad 0 \leq x,\, t \leq 1, \end{array} \tag{52}$$

with the boundary conditions $y(0) = 1$, $y(1) = 1$. Under such circumstances, the followings can be produced:

$$\begin{gathered} p_{10}(x) = 2x,\ \ p_{20}(x) = 1,\ \ p_{21}(x) = 1, \\ \mu_{10} = 1,\ \ \tau_{10} = -2,\ \ \mu_{20} = 1,\ \ \tau_{20} = 0,\ \ \mu_{21} = 1,\ \ \tau_{21} = -1, \\ \gamma_{00} = 1,\ \ \delta_{00} = 0,\ \ g(x) = 4x^2 - 15x + 4,\ \ \lambda = 12,\ \ \kappa_{00}(x,\, t) = x\, t. \end{gathered} \tag{53}$$

The exact solution of this example is $y(x) = x^2 - x + 1$. Let us assume that the problem has a B-polynomial solution in the form

$$y(x) = \sum_{i=0}^{n} c_i\, B_{i,\, n}(x),\ \ 0 \le x \le 1. \tag{54}$$

By applying the similar technique to what was described in the section 3, the matrix representation equation of this problem can therefore, be expressed as follow:

$$\begin{aligned} \big\{ P_{10}\, T\, Z(1,\, -2)\, D_{T_n} + P_{20}\, T\, Z(1,\, 0)\, D_{T_n}^2 + P_{21}\, T\, Z(1,\, -1)\, D_{T_n}^2 \\ -12\, T\, A\, K_{00}\, A^T H\, Z(1,\, 0) \big\}\, AC = G. \end{aligned} \tag{55}$$

Here, for $n = 2$ the collocation points are computed as

$$x_0 = 0,\ \ x_1 = \frac{1}{2},\ \ x_2 = 1, \tag{56}$$

and we have

$$\begin{gathered} P_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ P_{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ P_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},\ Z(1,\, -2) = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix},\ Z(1,\, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ Z(1,\, -1) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix},\ D_{T_n} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \\ K_{00} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix},\ H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix},\ G = \begin{bmatrix} 4 \\ -\frac{5}{2} \\ -7 \end{bmatrix}. \end{gathered} \tag{57}$$

For the boundary conditions $y(0) = 1$ and $y(1) = 1$, the augmented matrices are respectively obtained in the following form:

$$[U_0; \lambda_0] = \begin{bmatrix} 1 & 0 & 0; & 1 \end{bmatrix}\ \ \mathit{and}\ \ [U_1; \lambda_1] = \begin{bmatrix} 0 & 0 & 1; & 1 \end{bmatrix}. \tag{58}$$

From (45), using the matrices P_{10}, P_{20}, P_{21}, T, $Z(1, -2)$, $Z(1, 0)$, $Z(1, -1)$, D_{T_n}, A, K_{00}, H, $[U_0; \lambda_0]$ and $[U_1; \lambda_1]$, the augmented matrix based on the conditions $y(0) = 1$ and $y(1) = 1$ can be found as:

$$[\bar{W};\, \bar{G}] = \begin{bmatrix} 4 & -8 & 4; & 4 \\ 1 & 0 & 0; & 1 \\ 0 & 0 & 1; & 1 \end{bmatrix}. \tag{59}$$

This system has the unique solution

$$C = \begin{bmatrix} 1, & \frac{1}{2}, & 1 \end{bmatrix}^T. \tag{60}$$

TABLE 1. Comparison of the absolute errors for Example 2.

x	$n=6$		$n=9$	
	Method of [25]	*Present method*	*Method of* [25]	*Present method*
0.0	0	0	0	0
0.2	$3.3357E-05$	$4.7276E-07$	$3.8135E-09$	$5.5864E-09$
0.4	$3.7219E-04$	$1.6717E-06$	$2.9036E-08$	$3.3643E-08$
0.6	$1.7152E-03$	$1.8788E-06$	$2.4991E-07$	$8.5778E-08$
0.8	$5.4450E-03$	$2.0250E-06$	$3.6276E-06$	$1.5776E-07$
1.0	$1.3998E-02$	$3.3795E-05$	$2.2228E-05$	$2.4457E-07$

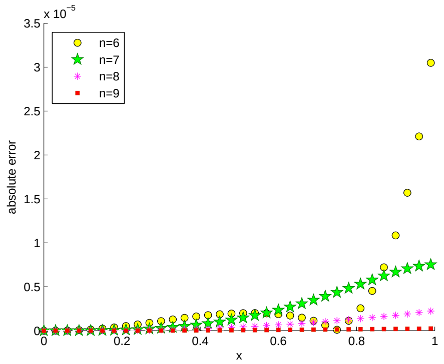

FIGURE 1. The comparison between absolute errors functions of Example 2 for some n.

By substituting the elements of this matrix into Eq. (3), the approximate solution of the given problem in terms of the B-polynomials of degree 2 can be described by the following relation:

$$y(x) = x^2 - x + 1, \tag{61}$$

which is the exact solution.

Example 2. Consider the third order linear Fredholm integro-differential-difference equation given in [15, 25, 50] by

$$\begin{aligned} &y'''(x) - (x-1)y''(x) + (x-1)y'(x) - y(x) + y'(x-1) \\ &= e^{x-1} + x\left(e\,x - \tfrac{1}{e}x - 2\tfrac{1}{e}\right) + \int_{-1}^{1} (x\,t - x^2)\,y(t)\,dt, \\ &-1 \le x,\, t \le 1, \end{aligned} \tag{62}$$

with the initial conditions $y(0) = y'(0) = y''(0) = 1$. The exact solution of which is $y(x) = e^x$. As can be seen from Table 1, the absolute errors obtained by the B-polynomials are compared with the absolute errors of the Fibonacci polynomials [25]. It can therefore, be observed that the yielded result is superior than the one provided [25]. The comparison between the absolute errors functions for $n = 6, 7, 8, 9$ can be seen in Figure 1.

Example 3. Consider the third order linear Fredholm integro-differential-difference equation given in [15, 52, 50] by

$$\begin{aligned} &y'''(x) - y'(x) = 2x\,(\cos 1 - \sin 1) - 2\,\cos(x) \\ &+\int_{-1}^{1} x\,t\,y(t)\,dt, \quad -1 \le x,\, t \le 1, \end{aligned} \tag{63}$$

under the mixed conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) - 2y'(0) = -2$. The exact solution is $y(x) = \sin x$. The absolute errors for $n = 9$ are obtained, the values of which are then tabulated for the

Table 2. Exact and approximate solutions for Example 3.

x	$Exact\,solution$	$Method\,of$ [52]	$Method\,of$ [15]	$Method\,of$ [50]	$Present\,method$
0.0	0	$-1.992188E-17$	$0.600000E-18$	$1.600000E-14$	$3.436446E-17$
0.2	0.19866933080	0.1986693309	0.1986693306	0.1986693275	0.1986693308
0.4	0.38941834231	0.3894183530	0.3894183416	0.3894183266	0.3894183422
0.6	0.56464247340	0.5646424233	0.5646424696	0.5646424405	0.5646424729
0.8	0.71735609090	0.7173547372	0.7173560843	0.7173560459	0.7173560894
1.0	0.84147098481	0.8414714504	0.8414709745	0.8414709465	0.8414709818

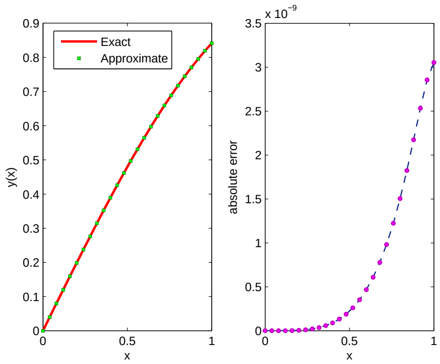

Figure 2. The comparison between approximate and exact solutions and absolute errors functions of Example 3 for $n = 9$.

Legendre [52], Chebyshev [15], Boubaker [50] and B-polynomials in Table 2. The superiority of proposed method in comparason with those presented in the methods [15, 52, 50] is vividly clear. This is because under the circumstances of having the same number of base functions, the absolute errors in the proposed method are quite lower. A plot of the exact and approximate solutions and absolute errors functions of this example for $n = 9$ are shown in Figure 2, which demonstrates the accuracy in the order of 10^{-9}.

Example 4. Consider the third order linear Fredholm integro-differential-difference equation given in [17, 25] by

$$\begin{aligned} y'''(x) - x\,y'(x) + y''(x-1) - x\,y(x-1) = &-(x+1)\,[\sin(x-1) + \cos x] - \cos 2 + 1 \\ &+ \int_{-1}^{1} y(t-1)\,dt, \quad -1 \le x,\, t \le 1, \end{aligned} \tag{64}$$

under the initial conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$. The exact solution is $y(x) = \sin x$. The problem is solved using the method described in Section 3 with $n = 6, 7, 8, 9$. The comparison of approximate solutions of the present method with that in [17] and the exact solution can be seen in Table 3. The compassion of the absolute errors for equal basis functions for the B-polynomials and Fibonacci polynomials [25] are shown in Table 4, showing the superiority of the yielded results by the present method relative to those in [17, 25]. Figure 3 shows a plot of Fibonacci method, Bernstein method and exact solution for $n = 8$.

Example 5. Consider the third order linear Fredholm integro-differential-difference equation given in [9] by:

$$\begin{aligned} &y'''(x) - x\,y''(x - \tfrac{\pi}{2}) - y'(x + \tfrac{\pi}{2}) = (x+1)\,\sin x \\ &+ \cos x - 2\,x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[x\,y'(t - \tfrac{\pi}{2}) - t\,y(t) - t\,y'(t + \tfrac{\pi}{2})\right]\,dt, \end{aligned} \tag{65}$$

TABLE 3. Exact and approximate solutions for Example 4.

x	*Exact solution*	$n=6$		$n=7$	
		Method of [17]	*Present method*	*Method of* [17]	*Present method*
0.0	0	0.000000	$1.34137E-16$	0.000000	$5.87294E-16$
0.2	0.198669	0.199128	0.198578	0.198616	0.198658
0.4	0.389418	0.393117	0.388822	0.388609	0.389347
0.6	0.564642	0.577468	0.563064	0.560822	0.564455
0.8	0.717356	0.749137	0.714589	0.705877	0.717030
1.0	0.841471	0.907265	0.837849	0.814098	0.841049

TABLE 4. Comparison of the absolute errors for Example 4.

x	$n=8$		$n=9$	
0.0	$5.6297E-17$	$4.3086E-16$	$1.1532E-15$	$2.9001E-15$
0.2	$2.9832E-03$	$3.5927E-05$	$1.6864E-02$	$5.1103E-08$
0.4	$2.4285E-02$	$2.3426E-04$	$1.3068E-01$	$3.2921E-07$
0.6	$8.3055E-02$	$6.1909E-04$	$4.2574E-01$	$8.5374E-07$
0.8	$1.9913E-01$	$1.0828E-03$	$9.7197E-01$	$1.4466E-06$
1.0	$3.9374E-01$	$1.4101E-03$	1.8261	$1.7696E-06$

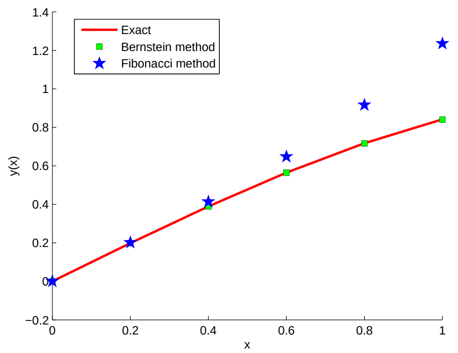

FIGURE 3. The comparison between exact and approximate solutions of Example 4 for $n=8$.

TABLE 5. Comparison of the absolute errors for Example 5.

x	$n=5$	$n=10$
0.0	0	$1.0658E-14$
0.2	$6.0008E-05$	$1.5446E-06$
0.4	$3.0581E-04$	$1.0662E-05$
0.6	$4.4703E-04$	$2.9754E-05$
0.8	$2.8758E-04$	$5.5130E-05$
1.0	$3.0546E-03$	$7.7492E-05$

with the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$, where the exact solution is $y(x) = \cos x$. This can be solved by application of the method described in Section 3 and

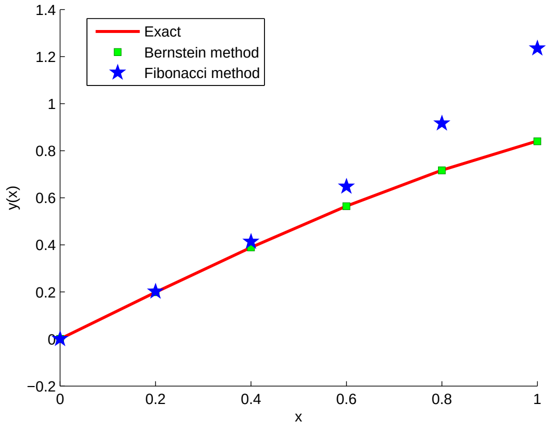

FIGURE 4. The comparison between exact and approximate solutions of Example 5 for $n = 10$.

the absolute errors for $n = 5, 10$ are tabulated in Table 5. These show that increase in the truncation limit n correspondingly increases the possibility of having less error. As can be observed with $n = 5, 10$ the absolute errors obtained by the application of Taylor polynomial method at $x = \pi/4$ are 1.3×10^{-2}, 1.8×10^{-4} respectively. However, the absolute errors at the same points in present method are 1.8×10^{-4}, 5.3×10^{-5} respectively. Figure 4 shows a plot of the approximate and exact solution of this example for $n = 10$, which shows almost identical solutions.

Example 6. Consider linear delay difference equation [19]

$$(x+3)\; y(x+2) - 2\,(x+2)\; y(x+1) + (x+1)\; y(x) = 0, \tag{66}$$

with the boundary conditions $y(0) = 0$, $y(1) = 1/2$, where the exact solution is $y(x) = x/(x+1)$. This can be solved by application of exact method as described in Section 3 with $n = 8, 9$. The comparison of absolute errors attributed to the present method and those in [19] is shown in Table 6, showing the B-polynomials having a higher accuracy than the Laguerre polynomials [19] (with the same degree).

TABLE 6. Comparison of the absolute errors for Example 6.

x	$n=8$		$n=9$	
	Method of [19]	*Present method*	*Method of* [19]	*Present method*
0.0	$3.2000E-06$	$5.3772E-14$	$1.0000E-07$	$7.9582E-14$
0.1	$6.1928E-04$	$1.9972E-04$	$5.5292E-04$	$5.4679E-04$
0.2	$2.3891E-04$	$5.8135E-04$	$9.1094E-04$	$8.3790E-04$
0.3	$1.5797E-03$	$9.6213E-04$	$9.2713E-04$	$8.5153E-04$
0.4	$2.7238E-03$	$1.1929E-03$	$6.9451E-04$	$6.4785E-04$
0.5	$3.3201E-03$	$1.2141E-03$	$3.6362E-04$	$3.3757E-04$
0.6	$3.2855E-03$	$1.0465E-03$	$6.1970E-05$	$3.6638E-05$
0.7	$2.7215E-03$	$7.6081E-04$	$1.3306E-04$	$1.6780E-04$
0.8	$1.8312E-03$	$4.4392E-04$	$1.9514E-04$	$2.3379E-04$
0.9	$8.4832E-04$	$1.7316E-04$	$1.3786E-04$	$1.6498E-04$
1.0	$1.7300E-05$	0	$6.9000E-07$	0

6. Conclusion

A numerical method for solving m -th order linear Fredholm integro-differential-difference equations under mixed conditions by approximating the solution in the Bernstein polynomial basis is proposed. The solution of equation was considered primarily in the form of expansion of Bernstein basis functions of n -th degree. The method proposed in this paper, reduces the high order linear Fredholm integro-differential-difference equations into a set of algebraic ones. One of the most important features of this method, among others, is that the Bernstein polynomial coefficients of the solution are found very easily by using computer programs. Another advantage being that under the circumstances where the solution of equation is in the form of a polynomial of degree equal to or less than n, then, the exact solution is obtained. However, in the absence of polynomial solution, Bernstein series approximation converges to the exact solution as n increases. Comparative assessment of applying the proposed method on six different test problems with their exact solutions and those of the existing methods shows the effectiveness and convenience of the proposed method. The method also demonstrates a high relative accuracy for small values of n, specially with a short run time. There is also a direct relation between the value of n and accuracy of the obtained result. In other words, any increment in the value of n would correspondingly increase the accuracy of the results. The approach can be applied with a slight modification to solve the system of linear integro-differential-difference equations having variable coefficients.

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